MAGIC CUBES.

THE curious and interesting characteristics of magic squares may be applied to figures of three dimensions constituting magic cubes.

Cubes of odd numbers may be constructed by direct and continuous process, and cubes of even numbers may be built up by the aid of geometrical diagrams. In each case the constructive methods resemble those which were explained in a previous article in connection with odd and even magic squares.

CHARACTERISTICS OF MAGIC CUBES.

The characteristics of magic cubes, odd or even, are that all straight columns, whether running from the top of the cube to the bottom, from the front to the back, or from one side to the other, should sum up to the same amount, also that the four diagonal columns which unite the eight corners of the cube and the two corner diagonal columns of every square in the cube should sum up to the same amount as the straight columns. Furthermore, the sum of any two numbers that are located in cells diametrically opposite to each other and equidistant from the center of the cube should equal the sum of the first and last numbers of the series used, and in all odd magic cubes, the center cell must contain the middle number of the series. Magic cubes which do not exhibit these characteristics may be considered imperfect.

Using N to express the number of cells in one column of a cube, which begins with unity and proceeds with increments of 1, the sum of the numbers in each column is expressed by the formula:

$$\frac{N}{2}(1+N^3)$$

If the initial and increment numbers are more or less than unity, the following general formula may be used to express the column value.

Let:

A = initial number,
B = increment number,
N = number of cells in each column,
S = summation number,

then:

$$\frac{N}{2}[B(N^3-1)+2A]=S.$$

ODD MAGIC CUBES.

The smallest magic cube is naturally $3 \times 3 \times 3$. The twenty-seven numbers in this cube are capable of many different magic arrangements, none of which, however, possess perfect characteristics.

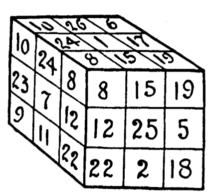

FIG. 1.

Fig. 1 shows one of these cubes, and in columns I, II and III, Fig. 2, there are given the nine different squares which it contains. In this cube there are twenty-seven straight columns, two diagonal columns in each of the

three middle squares, and four diagonal columns connecting the eight corners of the cube, making in all thirty-seven columns each of which sums up to 42. The center number is also 14 or $\frac{1+N^3}{2}$ and the sum of any pair of geometrically opposite numbers is 28 or $1+N^3$. In these points this cube approaches perfection, but it fails in the fact that the totals of the corner diagonals of the six outside squares consist of various numbers other than 42.

THREE SQUARES FROM TOP TO BOTTOM
COLUMN I.

10	26	6
24	1	17
8	15	19

23	3	16
7	14	21
12	25	5

9	13	20
11	27	4
22	2	18

THREE SQUARE FROM FRONT TO BACK
COLUMN II.

8	15	19
12	25	5
22	2	18

24	1	17
7	14	21
11	27	4

10	26	6
23	3	16
9	13	20

THREE SQUARES FROM LEFT TO RIGHT
COLUMN III.

10	24	8
23	7	12
9	11	22

26	1	15
3	14	25
13	27	2

6	17	19
16	21	5
20	4	18

FIG. 2.

All totals = 42.

In describing the direct method of building odd magic squares in a previous article, many forms of regular advance moves were explained, including right and left diagonal sequence, and various so-called "knights' moves." It was also shown that the order of regular advance was periodically broken by other well defined spacings which were termed "breakmoves." In building odd magic squares, only one form of breakmove was employed in each

square, but in the construction of odd magic cubes, two kinds are required in each cube which for distinction may be termed N and N^2 breakmoves respectively. In magic cubes which commence with unity and proceed with increments of 1, the N^2 breakmoves occur between each multiple of N^2 and the next following number, which in a $3 \times 3 \times 3$ cube brings them between 9 and 10, 18 and 19, and also between the first and last numbers of the series, 27 and 1. The N breakmoves are made between all other multiples of N, which in the above case brings them between 3 and 4, 6 and 7, 12 and 13, 15 and 16, 21 and 22, and 24 and 25. With this explanation the rules for building the magic cube shown in Fig. 1 may now be formulated, and for convenience of observation and construction, the cube is divided horizontally into three sections or layers, each section being shown separately in Column I, Fig. 2.

It may be mentioned that when a move is to be continued *upward* from the top square it is carried around to the bottom square, and when a move is to made *downward* from the bottom square, it is carried around to the top square, the conception being similar to that of the horizontal cylinder used in connection with odd magic squares.

Commencing with 1 in the center cell of the top square, the cells in the three squares are filled with consecutive numbers up to 27 in accordance with the following directions:

Advance move. One cell down in next square up (from last entry).

N breakmove. One cell in downward right hand diagonal in next square down (from last entry).

N^2 breakmove. Same cell in next square down (from last entry).

If it is desired to build this cube from the three vertical squares from front to back of Fig. 1, as shown in Column II, Fig. 2, the directions will then be as follows: commencing with 1 in the middle cell of the upper row of numbers in the middle square,

Advance move. One cell up in next square up.
N breakmove. One cell in downward right-hand diagonal in next square up.
N^2 breakmove. Same cell in next square down (from last entry).

TABLE I.

	A	B	C		A	B	C		A	B	C
1	1	1	1	10	2	1	1	19	3	1	1
2	1	1	2	11	2	1	2	20	3	1	2
3	1	1	3	12	2	1	3	21	3	1	3
4	1	2	1	13	2	2	1	22	3	2	1
5	1	2	2	14	2	2	2	23	3	2	2
6	1	2	3	15	2	2	3	24	3	2	3
7	1	3	1	16	2	3	1	25	3	3	1
8	1	3	2	17	2	3	2	26	3	3	2
9	1	3	3	18	2	3	3	27	3	3	3

FIG. 3.

Finally, the same cube may be constructed from the three vertical squares running from left to right side of Fig. 1, as shown in Column III, Fig. 2 commencing, as in the last example, with 1 in the middle cell of the upper row of numbers in the middle square, and proceeding as follows:

Advance move. Three consecutive cells in upward right-hand diagonal in same square (as last entry).

N breakmove. One cell in downward right-hand diagonal in next square down.

N^2 breakmove. One cell down in same square (as last entry).

Five variations may be derived from this cube in the simple way illustrated in Table I on the preceding page.

Assign three-figure values to the numbers 1 to 27 inclusive in terms of 1, 2, 3 as given in Table I, Fig. 3, and change the numbers in the three squares in Column I, Fig. 2, to their corresponding three-figure values, thus producing the square shown in Fig. 4. It is evident that if the arrangement of numbers in the three squares in Column I were unknown, they could be readily produced from Fig. 4 by the translation of the three-figure values into regular numbers in accordance with Table I, but more than this can be accomplished. The letters A, B, C in Table I in-

	A	B	C	A	B	C	A	B	C	
Top Square	2	1	1	3	3	2	1	2	3	1st Line
	3	2	3	1	1	1	2	3	2	2nd "
	1	3	2	2	2	3	3	1	1	3rd "
Middle Square	3	2	2	1	1	3	2	3	1	1st Line
	1	3	1	2	2	2	3	1	3	2nd "
	2	1	3	3	3	1	1	2	2	3rd "
Bottom Square	1	3	3	2	2	1	3	1	2	1st Line
	2	1	2	3	3	3	1	2	1	2nd "
	3	2	1	1	1	2	2	3	3	3rd "

FIG. 4.

dicate the normal order of the numerals 1, 2, 3, but by changing this order other triplets of 3×3 squares can be made which will differ more or less from the original models in the arangement of their cell numbers, but which will retain their general magic characteristics. The

changes which may be rung on A, B, C, are naturally six, as follows:

A. B. C.	C. B. A.
B. C. A.	B. A. C.
C. A. B.	A. C. B.

2	18	22
24	1	17
16	23	3

15	19	8
7	14	21
20	9	13

25	5	12
11	27	4
6	10	26

FIG. 5. (B. C. A.)

4	18	20
26	1	15
12	23	7

17	19	6
3	14	25
22	9	11

21	5	16
13	27	2
8	10	24

FIG. 6. (C. A. B.)

2	24	16
18	1	23
22	17	3

15	7	20
19	14	9
8	21	13

25	11	6
5	27	10
12	4	26

FIG. 7. (C. B. A.)

4	26	12
18	1	23
20	15	7

17	3	22
19	14	9
6	25	11

21	13	8
5	27	10
16	2	24

FIG. 8. (B. A. C.)

10	24	8
26	1	15
6	17	19

23	7	12
3	14	25
16	21	5

9	11	22
13	27	2
20	4	18

FIG. 9. (A. C. B.)

The combination of 1, 2, 3 being given in normal order in the original cube, the five cubes formed from the other combinations are shown in Figs. 5, 6, 7, 8, and 9.

These magic cubes may also be constructed by the direct method in accordance with the simple directions set forth in the adjoined diagram.

Fig 10 is an example of another $3 \times 3 \times 3$ cube in

TOP SQUARE.

1	17	24
15	19	8
26	6	10

MIDDLE SQUARE.

23	3	16
7	14	21
12	25	5

BOTTOM SQUARE.

18	22	2
20	9	13
4	11	27

FIG. 10.

which the first number occupies a corner cell, and the last number fills the diametrically opposite corner cell, the middle number coming in the center cell in accordance

DIRECTIONS FOR CONSTRUCTING THE 3×3×3 MAGIC CUBE SHOWN IN FIG. I, AND FIVE VARIATIONS OF THE SAME.

COMBINATION	ADVANCE MOVES	N BREAKMOVES	N^2 BREAKMOVES
A. B. C.	One cell down in next square up	One cell in right-hand downward diagonal in next square down	Same cell in next square down
B. C. A.	Three consecutive cells in upward left-hand diagonal in same square	One cell to left in next square up	Same as in A. B. C.
C. A. B.	One cell to right in next square up	One cell up in next square up	Same as in A. B. C.
C. B. A.	Same as in B. C. A.	Same as in C. A. B.	Same as in A. B. C.
B. A. C.	Same as in A. B. C.	Same as in B. C. A.	Same as in A. B. C.
A. C. B.	Same as in C. A. B.	Same as in A. B. C.	Same as in A. B. C.

with the rule. Fig. 11 shows this cube with the numbers changed to their three-figure values from which five variations of Fig. 10 may be derived, or they may be constructed

	A	B	C	A	B	C	A	B	C	
Top Square	1	1	1	2	3	2	3	2	3	1st Line
	2	2	3	3	1	1	1	3	2	2nd "
	3	3	2	1	2	3	2	1	1	3rd "
Middle Square.	3	2	2	1	1	3	2	3	1	1st Line
	1	3	1	2	2	2	3	1	3	2nd "
	2	1	3	3	3	1	1	2	2	3rd "
Bottom Square.	2	3	3	3	2	1	1	1	2	1st Line
	3	1	2	1	3	3	2	2	1	2nd "
	1	2	1	2	1	2	3	3	3	3rd "

FIG. 11.

directly by the directions which are marked in the diagram with the changes of A. B. C. for convenient reference.

The analysis of the numbers in Fig. 1 and Fig. 10 into their three-figure values in terms of 1, 2, 3, as shown in Figs. 4 and 11, makes clear the curious mathematical order of their arrangement which is not apparent on the face of the regular numbers as they appear in the various cells of the cubes. For example, it may be seen that in every subsquare in Figs. 4 and 11 (corresponding to horizontal columns in the cubes) the numbers 1, 2, 3 are each repeated three times. Also in every horizontal and perpendicular column there is the same triple repetition. Furthermore, all the diagonal columns in the cubes which sum up to 42, if followed into their analyses in Figs. 4 and 11 will also be found to carry similar repetitions. A brief study of these figures will also disclose other curious mathematical qualities pertaining to their intrinsic symmetrical arrangement.

The next odd magic cube in order is $5 \times 5 \times 5$, and Fig. 12 shows one of its many possible variations. For convenience, it is divided into five horizontal sections or

DIRECTIONS FOR CONSTRUCTING THE $3 \times 3 \times 3$ MAGIC CUBE SHOWN IN FIG. 10, AND FIVE VARIATIONS OF THE SAME.

COMBINATIONS	ADVANCE MOVES	N BREAKMOVES	N^2 BREAKMOVES
A. B. C.	One cell to left in next square up	One cell in upward left-hand diagonal in next square down	One cell in downward right-hand diagonal in next square down
B. C. A.	Three consecutive cells in upward left-hand diagonal in same square	One cell in upward right-hand diagonal in next square up	Same as in A. B. C.
C. A. B.	One cell up in next square up	One cell in downward left-hand diagonal in next square up	Same as in A. B. C.
C. B. A.	Same as in B. C. A.	Same as in C. A. B.	Same as in A. B. C.
B. A. C.	Same as in A. B. C	Same as in B. C. A.	Same as in A. B C.
A. C. B.	Same as in C. A, B.	Same as in A. B. C.	Same as in A B. C.

layers, forming five 5×5 squares from the top to the bottom of the cube.

Commencing with 1 in the first cell of the middle horizontal column in the third square, this cube may be constructed by filling in the various cells with consecutive numbers up to 125 in accordance with the following directions:

1.

67	98	104	10	36
110	11	42	73	79
48	54	85	111	17
86	117	23	29	60
4	35	61	92	123

TOP SQUARE.

3.

50	51	82	113	19
88	119	25	26	57
1	32	63	94	125
69	100	101	7	38
107	13	44	75	76

5.

3	34	65	91	122
66	97	103	9	40
109	15	41	72	78
47	53	84	115	16
90	116	22	28	59

BOTTOM SQUARE.

2.

106	12	43	74	80
49	55	81	112	18
87	118	24	30	56
5	31	62	93	124
68	99	105	6	37

4.

89	120	21	27	58
2	33	64	95	121
70	96	102	8	39
108	14	45	71	77
46	52	83	114	20

FIG. 12.

Advance moves. One cell up in next square down.

N breakmove. Two cells to the left and one cell down (knight's move) in same square as the last entry.

N^2 breakmove. One cell to right in same square as last entry.

Although this cube is not perfect it exhibits some interesting qualifications. Examining first the five horizontal squares from the top to the bottom of the cube as shown in Fig. 12, there are:

a. 50 straight columns summing up to...... 315
b. 10 corner diagonal columns summing up to 315
c. 40 sub-diagonal columns summing up to.. 315
Total 100 columns having the same summation.

In the five vertical squares from front to back of this cube there are:

a. 50 straight columns summing up to 315
b. 6 corner diagonal columns summing up to 315
c. 20 sub-diagonal columns summing up to.. 315
Total 76 columns having the same summation.

In the five vertical squares from right to left of cube, there are, as in the last case, 76 columns which all sum up to 315. In the complete cube there are also four diagonal columns which unite the eight corners that sum up to 315.

A table similar to Fig. 3 may be laid out giving three-figure values for the numbers in $5 \times 5 \times 5$ cubes from 1 to 125, and by changing the numbers in Fig, 12 to these

TABLE II.

Prime Nos.	1	2	3	4	5	Section 1.
Key Nos.	0	5	10	15	20	
Prime Nos.	1	2	3	4	5	Section 2.
Key Nos.	25	30	35	40	45	
Prime Nos.	1	2	3	4	5	Section 3.
Key Nos.	50	55	60	65	70	
Prime Nos.	1	2	3	4	5	Section 4.
Key Nos.	75	80	85	90	95	
Prime Nos.	1	2	3	4	5	Section 5.
Key Nos.	100	105	110	115	120	

FIG 13.

three-figure values, a square similar to Fig. 4 will be produced from which five variations of Fig. 12 may be derived. Similar results, however, can be obtained with less work by means of a table of key numbers constructed as shown in Fig. 13. (Table II.)

The three-figure values of cell numbers in $5 \times 5 \times 5$ magic cubes are found from this table as follows:

Select the key-number which is nearest to the cell-number, but *below it in value.* Then write down

1. The section number in which the key number is found,
2. The prime number over the key-number,

DIRECTIONS FOR CONSTRUCTING THE $5 \times 5 \times 5$ MAGIC CUBE SHOWN IN FIG. 12, AND FIVE VARIATIONS OF THE SAME.

COMBINATIONS	ADVANCE MOVES	N BREAKMOVES	N^2 BREAKMOVES
A. B. C.	One cell up in next square down	Two cells to left and one down in same square as last entry	One cell to right in same square as last entry
B. C. A.	Two cells to left and one up for five consecutive numbers in same square	Two cells in upward left hand diagonal in next square down	Same as in A. B. C.
C. A. B.	Two cells in left hand downward diagonal in next square up	One cell in right-hand downward diagonal in next square up	Same as in A. B. C.
C. B A,	Same as in B, C. A.	Same as in C. A. B.	Same as in A. B. C.
B. A. C.	Same as in A. B. C.	Same as in B. C. A.	Same as in A. B. C.
A. C. B.	Same as in C. A. B.	Same as in A. B. C.	Same as in A. B. C.

3. The difference between the key-number and the cell-number.

Three figures will thus be determined which will represent the required three-figure value of the cell-number.

Examples. The first number in the first row of the upper square in Fig. 12 is 67. The nearest key number to this and below it in value is 65 in section 3 under the prime number 4 and the difference between the key-number and the cell-number is 2. The three-number value of 67 is therefore 3. 4. 2. Again, the fourth number in the same row is 10. The nearest key-number but *below it in value* is 5 in section 1 under the prime number 2 and the difference between the key-number and the cell-number is 5. The three-figure value of 10 is therefore 1.2.5. By these simple operations the three-figure values of all the cell numbers in the $5 \times 5 \times 5$ cube in Fig. 12 may be quickly

1.

1	82	38	119	75
74	5	81	37	118
117	73	4	85	36
40	116	72	3	84
83	39	120	71	2

TOP SQUARE.

3.

65	16	97	28	109
108	64	20	96	27
26	107	63	19	100
99	30	106	62	18
17	98	29	110	61

5.

124	55	6	87	43
42	123	54	10	86
90	41	122	53	9
8	89	45	121	52
51	7	88	44	125

BOTTOM SQUARE.

2.

33	114	70	21	77
76	32	113	69	25
24	80	31	112	68
67	23	79	35	111
115	66	22	78	34

4.

92	48	104	60	11
15	91	47	103	59
58	14	95	46	102
101	57	13	94	50
49	105	56	12	93

FIG. 14.

determined, and by the system of transposition previously explained, five variations of this cube may be constructed.

The shorter method of building these $5 \times 5 \times 5$ cubes by the direct process of filling the different cells in regular order with consecutive numbers may, however, be considered by some to be preferable to the more roundabout way as seen in the table on page 400.

Fig. 14 is another example of a $5 \times 5 \times 5$ magic cube

DIRECTIONS FOR CONSTRUCTING THE $5 \times 5 \times 5$ MAGIC CUBE SHOWN IN FIG. 14, AND FIVE VARIATIONS OF THE SAME.

COMBINATIONS	ADVANCE MOVES	N BREAKMOVES	N^2 BREAKMOVES
A. B. C.	Two consecutive cells in upward left hand diagonal in next square up	One cell in upward right-hand diagonal in next square up	One cell in downward right-hand diagonal in next square down
B. C. A.	Two cells down in second square down	One cell in downward left-hand diagonal in second square down	Same as in A. B. C.
C. A. B.	Two cells to right in next square up	Two cells in downward right hand diagonal in next square down	Same as in A. B. C.
C. B. A.	Some as in B. C. A.	Same as in C. A. B.	Same as in A. B. C.
B. A. C.	Same as in A. B. C.	Same as in B. C. A.	Same as in A. B. C.
A. C. B.	Same as in C. A. B.	Same as in A. B. C.	Same as in A. B. C.

which is commenced in the upper left-hand corner of the top square, and finished in the lower right-hand corner of the bottom square, the middle number of the series (63) appearing in the center cell of the cube according to rule.

Odd magic cubes may be commenced in various cells other than those shown in the preceding pages, and they may be built up with an almost infinite number of variations. It would, however, be only superfluous and tiresome to amplify the subject further, as the examples already submitted cover all the important points of construction, and may readily be applied to further extensions.

Any sizes of odd magic cubes larger than $5 \times 5 \times 5$ may be constructed by the directions which govern the formation of $3 \times 3 \times 3$ and $5 \times 5 \times 5$ cubes.

EVEN MAGIC CUBES.

Magic cubes of even numbers may be built by the aid of geometric diagrams, similar to those illustrated in a previous article, which described the construction of even magic squares.

Fig. 15 shows one of the many possible arrangements of a $4 \times 4 \times 4$ cube, the diagram of which is given in Fig. 16.

There are fifty-two columns in this cube which sum up to 130, viz., sixteen vertical columns from the top of the cube to the bottom, sixteen horizontal columns from the front to the back, sixteen horizontal columns from right to left, and four diagonal columns uniting the four pairs of opposite corners. The sum of any two numbers, which are diametrically opposite to each other and equidistant from the center of the cube also equals 65 or n^3+1.

Another feature of this cube is that the sum of the four numbers in each of the forty-eight sub-squares of 2×2 is 130.

It was shown in a previous article on magic squares

(Top.) Section I.

1	63	62	4
60	6	7	57
56	10	11	53
13	51	50	16

Section II

48	18	19	45
21	43	42	24
25	39	38	28
36	30	31	33

Section III.

32	34	35	29
37	27	26	40
41	23	22	44
20	46	47	17

Section IV. (Bottom.)

49	15	14	52
12	54	55	9
8	58	59	5
61	3	2	64

FIG. 15.

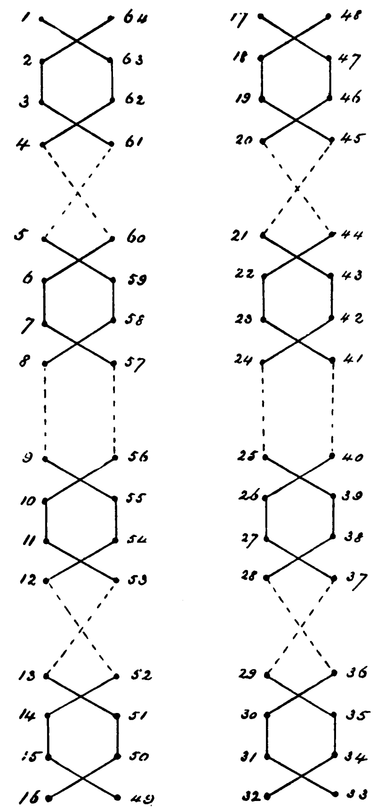

FIG. 16.

Totals = 130.

that the square of 4×4 could be formed by writing the numbers 1 to 16 in arithmetical order, then leaving the numbers in the two corner diagonals unchanged, but changing all the other numbers to their complements with 17 or n^2+1. It will be noted in the magic cube of 4×4×4, given in Fig. 15, that in the first and last of the four sections (I and IV) this rule also holds good. In the two middle sections (II and III) the rule is reversed; the numbers in the two corner diagonals being complements with

65 or n^3+1, and all the other numbers in arithmetical order.

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

17	18	19	20
21	22	23	24
25	26	27	28
29	30	31	32

33	34	35	36
37	38	39	40
41	42	43	44
45	46	47	48

49	50	51	52
53	54	55	56
57	58	59	60
61	62	63	64

FIG. 17.

Fig. 17 shows four squares or sections of a cube, with the numbers 1 to 64 written in arithmetical order. Those numbers that occupy corresponding cells in Fig. 15 are enclosed within circles. If all the other numbers in Fig. 17 are changed to their complements with 65, the total arrangement of numbers will then be the same as in Fig. 15.

In his interesting and instructive "Reflections on Magic Squares," (printed in the January number of *The Monist*) Dr. Paul Carus gives a novel and ingenious analysis of even squares in different "orders" of numbering, these orders being termed respectively *o, ro, i* and *ri*. It is shown that the two magic squares of 4×4 (in article referred to) consist only of *o* and *ro* numbers; *ro* numbers being in fact the complements of *o* numbers with n^2+1. This rule also obtains in the magic cube of 4×4×4 given in Fig. 15. The four sections of this cube may in fact be filled out by writing the *o* numbers, in arithmetical order in the cells of the two corner diagonal columns of sections I and IV, and in all the cells of sections II and III, excepting those of the two corner diagonal columns, and then writing the *ro* numbers, also in arithmetical order, in the remaining empty cells of the four sections.

Fig. 15 may be considered as the foundation of all magic cubes of 4×4×4 and their multiples, of this class,

but a great many variations may be effected by simple transpositions. For example, Fig. 18 is a $4 \times 4 \times 4$ cube which is constructed by writing the four numbers that are contained in the 2×2 sub-squares (Fig. 15) in a straight line, and there are many other possible transpositions which will change the relative order of the numbers, without destroying the magic characteristics of the cube.

The arrangement of the numbers in Fig. 18 follows the diagrammatic order shown in Fig. 19.

Section I. (Top.)

1	63	60	6
62	4	7	57
56	10	13	51
11	53	50	16

Section II.

48	18	21	43
19	45	42	24
25	39	36	30
38	28	31	33

Section III.

32	34	37	27
35	29	26	40
41	23	20	46
22	44	47	17

Section IV. (Bottom.)

49	15	12	54
14	52	55	9
8	58	61	3
59	5	2	64

FIG. 18.

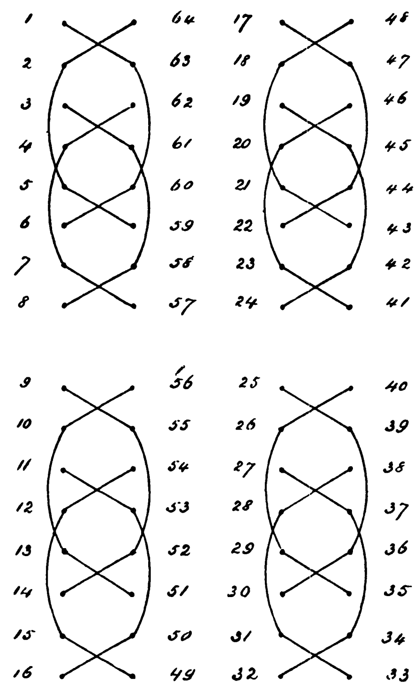

FIG. 19.

Totals = 130.

The next even magic cube is 6×6×6, but the writer has been unable to construct it. He does not however believe it to be a mathematical impossibility.

The 8×8×8 magic cube follows next in order. Fig. 20 shows this cube divided, for convenience, into eight horizontal layers or sections, and Fig. 21 gives the diagrammatic order of the numbers in the first and eighth sections, the intermediate sections being built from similar diagrams, numbered in arithmetical order.

It will be seen from these diagrams that the 8×8×8 magic cube is simply an expansion of the 4×4×4 cube,

1	511	510	4	5	507	506	8
504	10	11	501	500	14	15	497
496	18	19	493	492	22	23	489
25	487	486	28	29	483	482	32
33	479	478	36	37	475	474	40
472	42	43	469	468	46	47	465
464	50	51	461	460	54	55	457
57	455	454	60	61	451	450	64

Section I.

448	66	67	445	444	70	71	441
73	439	438	76	77	435	434	80
81	431	430	84	85	427	426	88
424	90	91	421	420	94	95	417
416	98	99	413	412	102	103	409
105	407	406	108	109	403	402	112
113	399	398	116	117	395	394	120
392	122	123	389	388	126	127	385

Section II.

384	130	131	381	380	134	135	377
137	375	374	140	141	371	370	144
145	367	366	148	149	363	362	152
360	154	155	357	356	158	159	353
352	162	163	349	348	166	167	345
169	343	342	172	173	339	338	176
177	335	334	180	181	331	330	184
328	186	187	325	324	190	191	321

Section III.

193	319	318	196	197	315	314	200
312	202	203	309	308	206	207	305
304	210	211	301	300	214	215	297
217	295	294	220	221	291	290	224
225	287	286	228	229	283	282	232
280	234	235	277	276	238	239	273
272	242	243	269	268	246	247	265
249	263	262	252	253	259	258	256

Section IV.

(First Part.)

FIG. 20.

just as the 8×8 magic square is an expansion of the 4×4 square. In like manner all the diagrams which were given for different arrangements of 8×8 magic squares may also be employed in the construction of various 8×8×8 magic cubes.

257	255	254	260	261	251	250	264
248	266	267	245	244	270	271	241
240	274	275	237	236	278	279	233
281	231	230	284	285	227	226	288
289	223	222	292	293	219	218	296
216	298	299	213	212	302	303	209
208	306	307	205	204	310	311	201
313	199	198	316	317	195	194	320

Section V.

128	386	387	125	124	390	391	121
393	119	118	396	397	115	114	400
401	111	110	404	405	107	106	408
104	410	411	101	100	414	415	97
96	418	419	93	92	422	423	89
425	87	86	428	429	83	82	432
433	79	78	436	437	75	74	440
72	442	443	69	68	446	447	65

Section VII.

192	322	323	189	188	326	327	185
329	183	182	332	333	179	178	336
337	175	174	340	341	171	170	344
168	346	347	165	164	350	351	161
160	354	355	157	156	358	359	153
361	151	150	364	365	147	146	368
369	143	142	372	373	139	138	376
136	378	379	133	132	382	383	129

Section VI.

449	63	62	452	453	59	58	456
56	458	459	53	52	462	463	49
48	466	467	45	44	470	471	41
473	39	38	476	477	35	34	480
481	31	30	484	485	27	26	488
24	490	491	21	20	494	495	17
16	498	499	13	12	502	503	9
505	7	6	508	509	3	2	512

Section VIII.

(Second Part.)
FIG. 20.

An examination of Fig. 20 will show that, like the 4×4×4 cube in Fig. 15 it is built up of *o* and *ro* numbers exclusively. In sections I, IV, V, and VIII, the cells in the corner diagonal columns, and in certain other cells which are placed in definite geometrical relations thereto,

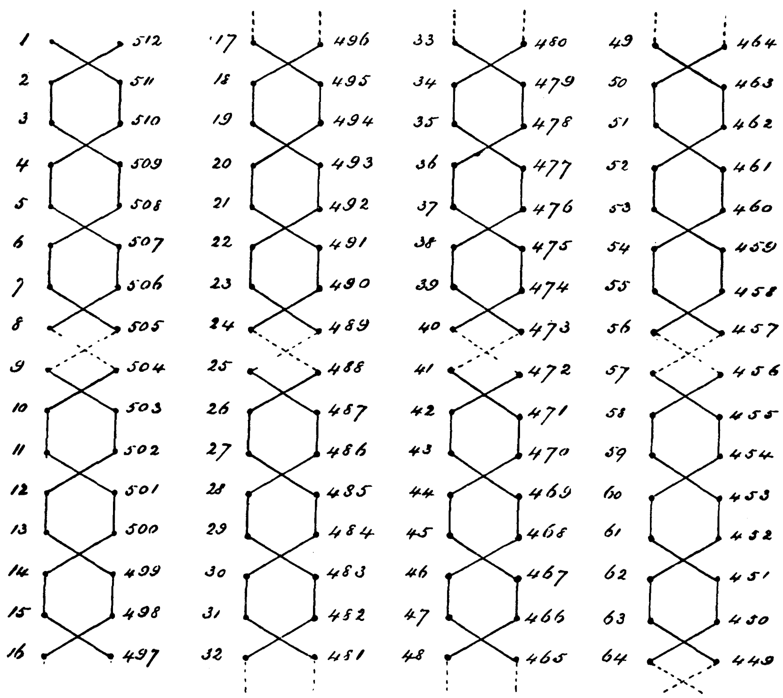

FIG. 21.

contain *o* numbers, while all the other cells contain *ro* numbers. In sections II, III, VI, and VII, the relative positions of the *o* and *ro* numbers are reversed.

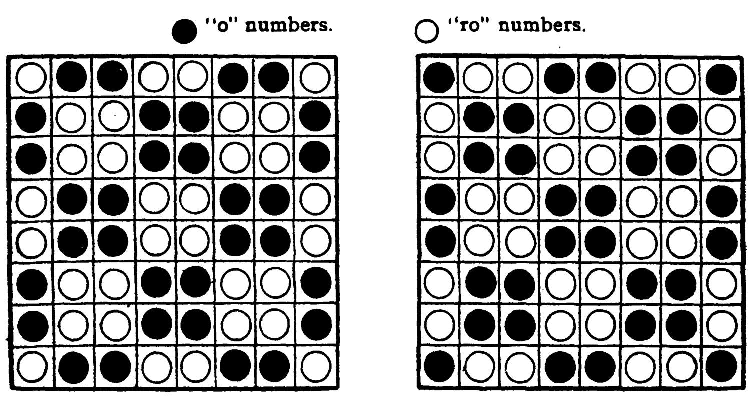

FIG. 22.

By noting the symmetrical disposition of these two orders of numbers in the different sections, the cube may be readily constructed without the aid of any geometrical diagrams. Fig. 22 shows sections I and II of Fig. 20 filled with *o* and *ro* symbols without regard to numerical values, and the relative symmetrical arrangement of the two orders is therein plainly illustrated. This clear and lucid analysis, for which we are indebted to Dr. Carus, reduces the formation of a rather complicated numerical structure to an operation of the utmost simplicity.

In this cube there are 192 straight columns, and 4 diagonal columns (which unite the eight corners of the cube) each of which sums up to 2052; also 384 half columns and the same number of 2×2 sub-squares each of which has the summation of 1026. It will also be seen that the sum of any two numbers, which are located in cells diametrically opposite to each other and equidistant from the center of the cube, is 513 or n^3+1.

As the writer has been unable to construct the $6\times6\times6$ magic cube no attempt will be made in this article to produce a $10\times10\times10$ or any larger cube of this peculiar class. The $12\times12\times12$ cube and all larger ones that are formed with multiples of 4 will naturally resemble the $8\times8\times8$ cube and will be equally easy to construct.

GENERAL NOTES ON MAGIC CUBES.

Magic cubes may be constructed having any desired summations by using suitable initial numbers with given increments, or by applying proper increments to given initial numbers.

* * *

The general formula for determining the summations or column values of magic cubes as already given on page 389 may be copied here for convenient reference.

Let A = initial number,
B = increment,
N = number of cells in each column of cube,
S = summation,

then:

I $$\frac{N}{2}\left[B(N^3-1)+2A\right]=S$$

and transposing:

II $$\frac{\dfrac{2S}{N}-2A}{N^3-1}=B.$$

or:

III $$\frac{\dfrac{2S}{N}-B(B^3-1)}{2}=A$$

EXAMPLES.

What increment number is required for the cube of $3\times3\times3$ with an initial number of 10 to produce summations of 108?

Expressing equation II in figure values:

$$\frac{\dfrac{2\times108}{3}-2\times10}{3^3-1}=2.$$

28	16	20
56	10	42
24	38	46

54	14	40
22	36	50
32	58	18

26	34	48
30	62	16
52	12	44

What increments should be used in a cube of $4\times4\times4$ to produce summations of 704 if the initial number is 50?

$$\frac{\dfrac{2\times704}{4}-2\times50}{4^3-1}=4.$$

Section I (Top).

50	298	294	62
286	70	74	274
270	86	90	258
98	250	246	110

Section II.

238	118	122	226
130	218	214	142
146	202	198	158
190	166	170	178

Section III.

174	182	186	162
194	154	150	206
210	138	134	222
126	230	234	114

Section IV (Bottom)

242	106	102	254
94	262	266	82
98	278	282	66
290	58	54	302

What initial number must be used with increments of 10 to produce summations of 1906 in a $3\times3\times3$cube?

Expressing equation III in figure values:

$$\frac{\frac{2\times1906}{3}-10(3^3-1)}{2}=505^1/_3.$$

Top Section.

$593\frac{1}{3}$	$755\frac{1}{3}$	$555\frac{1}{3}$
$735\frac{1}{3}$	$505\frac{1}{3}$	$665\frac{1}{3}$
$575\frac{1}{3}$	$645\frac{1}{3}$	$685\frac{1}{3}$

Middle Section.

$725\frac{1}{3}$	$525\frac{1}{3}$	$655\frac{1}{3}$
$565\frac{1}{3}$	$635\frac{1}{3}$	$705\frac{1}{3}$
$615\frac{1}{3}$	$745\frac{1}{3}$	$545\frac{1}{3}$

Bottom Section.

$585\frac{1}{3}$	$625\frac{1}{3}$	$695\frac{1}{3}$
$605\frac{1}{3}$	$765\frac{1}{3}$	$535\frac{1}{3}$
$715\frac{1}{3}$	$515\frac{1}{3}$	$675\frac{1}{3}$

What initial number is required for the cube of $5\times5\times5$, with 4 as increment number, to produce summations of 1906?*

$$\frac{\frac{2\times1906}{5}-4(5^3-1)}{2}=133.2$$

* * *

The preceding simple examples will be sufficient to illustrate the formulæ given, and may suggest other problems to those who are interested in the subject.

It will be noted that the magic cubes which have been

* This example was contributed by Mr. D. B. Ventres of Haddam, Conn., whom the writer takes this opportunity to thank for many interesting suggestions and ideas.

described in this article are all in the same general class as the magic squares which formed the subject of previous articles.

397.2	521.2	545.2	169.2	273.2
569.2	173.2	297.2	421.2	445.2
321.2	345.2	469.2	573.2	197.2
473.2	597.2	221.2	245.2	369.2
145.2	269.2	373.2	497.2	621.2

Section I (Top).

553.2	177.2	301.2	425.2	449.2
325.2	349.2	453.2	577.2	201.2
477.2	601.2	225.2	249.2	353.2
149.2	253.2	377.2	501.2	625.2
401.2	525.2	549.2	153.2	277.2

Section II.

329.2	333.2	457.2	581.2	205.2
481.2	605.2	229.2	233.2	357.2
133.2	257.2	381.2	505.2	629.2
405.2	529.2	533.2	157.2	281.2
557.2	181.2	305.2	429.2	433.2

Section III.

485.2	609.2	213.2	237.2	361.2
137.2	261.2	385.2	509.2	613.2
409.2	513.2	537.2	161.2	285.2
561.2	185.2	309.2	413.2	437.2
313.2	337.2	461.2	585.2	209.2

Section IV.

141.2	265.2	389.2	493.2	617.2
393.2	517.2	541.2	165.2	289.2
565.2	189.2	293.2	417.2	441.2
317.2	341.2	465.2	589.2	193.2
469.2	593.2	217.2	241.2	365.2

Section V.

There are, however, many classes of magic squares and corresponding cubes which differ from these in the general

arrangement of numbers and in various minor features, while retaining the common characteristic of having similar column values. An example of this differentiation is seen in the interesting "Jaina" square described by Dr. Carus in his "Reflections on Magic Squares." Squares of this class can readily be expanded into cubes which will naturally carry with them the peculiar features of the squares.

Another class is illustrated in the "Franklin Squares," and these can also be expanded into cubes constructed on the same general principles.

The subject of magic squares and cubes is indeed inexhaustible and may be indefinitely extended. The philosophical significance of these studies has been so ably set forth by Dr. Carus that the writer feels unable to add anything in this connection, but he trusts that the present endeavor to popularize these interesting problems may some time lead to useful results.

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